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On the Vepstas Representation for the Riemann Zeta Function

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Abstract: We show that the Vepstas expression for the Riemann zeta function implies interesting relations between Stirling, Harmonic and Bernoulli numbers and the Stieltjes constants.

Key words: Riemann zeta function - Stirling numbers - Stieltjes constants - Bernoulli numbers

INTRODUCTION

Vepstas [1] obtained the following representation for the Riemann zeta function [2]:

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} - s \sum_{k=0}^{\infty} (-1)^k \binom{s-1}{k} a_k, s \neq 1, \tag{1}$$

which gives the value $\zeta(0) = -\frac{1}{2}$. On the other hand, we have the expansion [2]:

$$\zeta(s) = \frac{1}{s-1} + \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \gamma_r (s-1)^r, s \neq 1, \tag{2}$$

where the γ_r are the Stieltjes constants; from (2) for $s = 0$:

$$\sum_{r=0}^{\infty} \frac{\gamma_{2r+1}}{(2r+1)!} = \frac{1}{2} - \sum_{r=0}^{\infty} \frac{\gamma_{2r}}{(2r)!}. \tag{3}$$

Now (2) for $s = 2$ implies the relation:

$$\sum_{r=0}^{\infty} \frac{\gamma_{2r}}{(2r)!} = \zeta(2) - 1 + \sum_{r=0}^{\infty} \frac{\gamma_{2r+1}}{(2r+1)!},$$

where we can apply (3) to deduce the properties:

$$\sum_{r=0}^{\infty} \frac{\gamma_{2r}}{(2r)!} = \frac{1}{2}\zeta(2) - \frac{1}{4} = \frac{\pi^2}{12} - \frac{1}{4}, \sum_{r=0}^{\infty} \frac{\gamma_{2r+1}}{(2r+1)!} = \frac{3}{4} - \frac{1}{2}\zeta(2) = \frac{3}{4} - \frac{\pi^2}{12}. \tag{4}$$

We know the following connection between a binomial coefficient and the Stirling numbers of the first kind [2-5]:

$$\binom{s-1}{k} = \frac{1}{k!} \sum_{r=0}^{k} S_k^{(r)} (s-1)^r, \tag{5}$$

Then (1) acquires the structure:

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \sum_{r=0}^{\infty} q_r (s-1)^r, \tag{6}$$

Such that:

$$q_r \equiv \sum_{k=r}^{\infty} \frac{(-1)^k}{k!} a_k S_k^{(r)} \therefore q_0 = a_0. \tag{7}$$

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The comparison of (2) and (6) gives the value:

$$\gamma_0 = \frac{1}{2} - q_0 \ \therefore \ a_0 = \frac{1}{2} - \gamma_0 , \tag{8}$$

where $\gamma_0 = 0.5772\,1566\,4901 \ldots$ is the famous Euler-Mascheroni's constant [2, 6-9], thus from (1) for $s = 2, 3, 4, \ldots$ [1]:

$$a_1 = \frac{1}{2}\,\zeta(2) - \gamma_0 - \frac{1}{4}\,, a_2 = \zeta(2) - \frac{1}{3}(\zeta(3) + 2) - \gamma_0, a_3 = \frac{3}{2}\zeta(2) - \zeta(3) + \frac{1}{4}\zeta(4) - \gamma_0 - \frac{23}{24}\,, \ldots \tag{9}$$

with the recurrence relation:

$$q_{n+1} + q_n = \frac{(-1)^n}{(n+1)!}\,\gamma_{n+1}\,, n \geq 0, \tag{10}$$

That is:

$$q_0 = -\gamma_0 + \frac{1}{2}\,, q_1 = \gamma_1 + \gamma_0 - \frac{1}{2}\,, q_2 = -\frac{1}{2}\gamma_2 - \gamma_1 - \gamma_0 + \frac{1}{2}\,, \ldots ; \tag{11}$$

The general solution of (10) is given by:

$$q_n = (-1)^n \left[\frac{1}{2} - \sum_{r=0}^{n} \frac{\gamma_r}{r!}\right], n \geq 0. \tag{12}$$

It is immediate the inversion of (7):

$$a_r = (-1)^r\, r!\, \sum_{k=r}^{\infty} q_k\, S_k^{[r]}\,, r \geq 0, \tag{13}$$

Involving Stirling numbers of the second kind [2-5]. We know [2, 3] that $S_k^{[1]} = 1, k \geq 1$, then from (13):

$$a_1 = -\sum_{k=1}^{\infty} q_k = -[(q_2 + q_1) + (q_4 + q_3) + (q_6 + q_5) + \cdots],$$

$$\overset{(10)}{=} \frac{\gamma_2}{2!} + \frac{\gamma_4}{4!} + \frac{\gamma_6}{6!} + \cdots = \sum_{r=0}^{\infty} \frac{\gamma_{2r}}{(2r)!} - \gamma_0 \overset{(4)}{=} \frac{1}{2}\zeta(2) - \gamma_0 - \frac{1}{4},$$

In agreement with (9).

Vepstas [1] deduced the values:

$$\sum_{r=0}^{\infty} a_r = \ln\!\left(\sqrt{2\pi}\right) - 1\,, \sum_{r=0}^{\infty} \frac{a_r}{2^r} = 2 - 3\ln 2\,, \tag{14}$$

Thus (13) and (14) imply the relations:

$$\sum_{k=0}^{\infty} (-1)^k\, q_k = \ln\!\left(\sqrt{2\pi}\right) - 1 = -0.0810\,6146\,6795 \ldots, \tag{15}$$

$$\sum_{k=0}^{\infty} \frac{1 - 2^{k+1}}{k+1}\, q_k\, B_{k+1} = 1 - 3\, ln\sqrt{2} = -0.0397\,2077\,084 \ldots,$$

Involving Bernoulli numbers [2-5]; in the deduction of (15) were applied the identities [2, 3, 10]:

$$\sum_{r=0}^{k} (-1)^r\ r!\ S_k^{[r]} = (-1)^k\,, \sum_{r=0}^{k} \frac{(-1)^r}{2^r}\ r!\ S_k^{[r]} = \frac{2\,(1 - 2^{k+1})}{k+1}\, B_{k+1}\,. \tag{16}$$

Now we use (7) for $r = 1, 2$ and (11) to obtain the expressions:

$$\sum_{k=1}^{\infty} \frac{a_k}{k} = \frac{1}{2} - \gamma_0 - \gamma_1 \, , \sum_{k=2}^{\infty} \frac{a_k}{k} \, H_{k-1} = \frac{1}{2} - \gamma_0 - \gamma_1 - \frac{1}{2} \, \gamma_2 \, , \quad (17)$$

Involving Harmonic numbers, where we employ the values:

$$S_k^{(1)} = (-1)^{k-1} \, (k-1)! \, , S_k^{(2)} = (-1)^k \, (k-1)! \; H_{k-1} \, . \quad (18)$$

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